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Søren R. K. Nielsen[1] and H. Uğur Köylüoğlu[2]

ABSTRACT

Path integration (cell-to-cell mapping) method is applied to evaluate the joint probability density function (jpdf) of the response of structural systems, with uncertain properties, subject to white noise excitation. A general methodology to deal with uncertainties is outlined and applied to the friction controlled slip of a structure on a foundation where the friction coefficient is modelled as a random variable. Exact results derived using the total probability theorem are compared to the ones obtained via path integration.

INTRODUCTION

Path integration is an approximate method to solve the Fokker-Planck equation of nonlinear dynamic systems driven by excitations with independent increments. Path integration methods have been applied to nonlinear dynamic systems subject to white noise by Sun and Hsu (1990), Naess and Johnsen (1991), Poisson driven train of impulses Köylüoğlu et al. (1995.a). In all of these studies, very accurate results are reported and it is pointed out that the path integration methods can only be applied to problems of low dimensionaliy due to the large memory and cpu time requirements of the numerical calculations.

In this paper, new equations are introduced to represent the random variables that model the uncertain properties. A deterministic system equation but with random initial conditions is then obtained for the integrated state vector made up of the displacement, velocity and random variables. Assuming that the random parameters are stochastically independent of the white noise excitation, the integrated state vector becomes Markovian, which further simplifies the problem. Then, the Chapman-Kolmogorov equation is discretized in space and time, and the Markov process is represented by a Markov chain. The proposed method has been applied to the friction controlled slip of a structure on a foundation where the friction coefficient is modelled as a random variable. The transitional probability matrix is calculated numerically using Gaussian closure applied for a small time step.

2. DEALING WITH UNCERTAIN PROPERTIES

The equation of motion of a SDOF system with mass m and random stiffness and damping subject to Gaussian white noise $W(t)$ with intensity S_0 reads in state vector

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form as

$$\dot{\mathbf{Z}}(t) = \mathbf{a}(\mathbf{Z}(t)) + \mathbf{b}(t)W(t) \quad , \quad \mathbf{Z}(0) = \mathbf{Z}_0 \tag{1}$$

The state vector $\mathbf{Z}(t)$, drift-vector $\mathbf{a}(\mathbf{Z}(t))$ and diffusion vector $\mathbf{b}(t)$ are given as

$$\dot{\mathbf{Z}}(t) = \begin{bmatrix} Y(t) \\ \dot{Y}(t) \\ \mathbf{V}(t) \end{bmatrix} \;,\; \mathbf{Z}_0 = \begin{bmatrix} 0 \\ 0 \\ \mathbf{X} \end{bmatrix} \;,\; \mathbf{a}(\mathbf{Z}(t)) = \begin{bmatrix} \dot{Y}(t) \\ -\frac{1}{m}H(Y,\dot{Y},\mathbf{V}) \\ \mathbf{0} \end{bmatrix} \;,\; \mathbf{b}(t) = \begin{bmatrix} 0 \\ \frac{1}{m} \\ \mathbf{0} \end{bmatrix} \tag{2}$$

where $H(Y,\dot{Y},\mathbf{X})$ is the nonlinear restoring force depending on the displacement $Y(t)$, velocity $\dot{Y}(t)$ and mean-zero random variables $\mathbf{X}^T = [X_1, X_2, \cdots X_d]$ specifying the stiffness and damping properties. $\mathbf{X}$ is assumed to be stochastically independent of the white noise excitation $W(t)$, and the jpdf $f_{\mathbf{X}}(\mathbf{x})$ is known. In (1), the random parameters have been introduced as extra state variables $\mathbf{V}(t)$ by the equation $\dot{\mathbf{V}}(t) = \mathbf{0}$, $\mathbf{V}(0) = \mathbf{X}$. Then, the equation of motion with random parameters $\mathbf{X}$ and deterministic initial conditions $Y(0) = \dot{Y}(0) = 0$ has been transformed into an equivalent system with deterministic coefficients and random initial conditions. Due to the white noise excitation and as $\mathbf{Z}_0$ is independent of $W(t)$, the state vector $\mathbf{Z}(t)$ of dimension $N = 2 + d$ is Markovian. Such a treatment has been investigated in Köylüoğlu et al. (1995.b).

3. PATH INTEGRATION

Let $q_{\mathbf{Z}}(\mathbf{z}, t_i|\mathbf{z}_0, t_{i-1})$ signify the transitional jpdf of the Markov process from the state $\mathbf{z}_0$ at time t_{i-1} to a state $\mathbf{z}$ at a later time t_i after the ith transition. The jpdf $f_{\mathbf{Z}}(\mathbf{z}, t_i)$ of $\mathbf{Z}(t_i)$ at time t_i then fulfills the Chapman-Kolmogorov equation :

$$f_{\mathbf{Z}}(\mathbf{z}, t_i) = \int_{\mathbf{z}_0} q_{\mathbf{Z}}(\mathbf{z}, t_i|\mathbf{z}_0, t_{i-1}) f_{\mathbf{Z}}(\mathbf{z}_0, t_{i-1}) d\mathbf{z}_0 \quad , \quad i = 1, 2, \ldots \quad , \quad t_0 = 0 \tag{3}$$

The initial distribution at time $t = 0$ is given as $f_{\mathbf{Z}}(\mathbf{z}, 0) = f_{Y\dot{Y}\mathbf{V}}(y, \dot{y}, \mathbf{v}, 0) = \delta(y)\delta(\dot{y})f_{\mathbf{X}}(\mathbf{v})$. Path integration demands a suitable discretization of the state space, and constant time intervals $\Delta t = t_i - t_{i-1}$ between the transitions. Then, the time and space continuous Markov vector $\mathbf{Z}(t)$ is carried into a Markov chain with the transition equation

$$\mathbf{p}_i = \mathbf{Q}\mathbf{p}_{i-1} \tag{4}$$

Using a uniform mesh, the jth component $p_{i,j} = \int_{\Delta \mathbf{z}_j} f_{\mathbf{Z}}(\mathbf{z}, t_i) d\mathbf{z}$ of $\mathbf{p}_i$ of (4) being the probability lumped to the jth nodal point with position $\mathbf{z}_j$, centered in a cell $\Delta \mathbf{z}_j = \Delta z_1 \cdots \Delta z_N$. Δz_l denotes the spacings of the mesh in the lth coordinate direction. The component $Q_{jk} = \int_{\Delta \mathbf{z}_j} q_{\mathbf{Z}}(\mathbf{z}, t_0 + \Delta t|\mathbf{z}_k, t_0) d\mathbf{z}$ of the transition probability matrix $\mathbf{Q}$ signifies the probability of being in node j at the time $t = t_0 + \Delta t$ on condition of starting at the kth node at time $t = t_0$. Resuming this procedure a number n of times, the state $\mathbf{p}_n$ is given as $\mathbf{p}_n = \mathbf{Q}^n \mathbf{p}_0$, attaining a stationary state as $n \to \infty$.

4. FRICTION CONTROLLED SLIP OF A STRUCTURE ON A FOUNDATION

The equation of motion for the slip displacement $S(t)$ of a structure on a foundation is

$$\ddot{S}(t) + \nu g \ \mathrm{sign}(\dot{S}) = -\ddot{G}(t) \tag{5}$$

where ν is the friction coefficient and $\ddot{G}(t)$ is the horizontal surface acceleration. In what follows, $\ddot{G}(t)$ is assumed to be white noise with intensity S_0. Then, the exact stationary conditional pdf and the variance for the slip velocity conditioned on ν are

$$f_{\dot{S}}(\dot{s}|\nu) = \frac{\nu g}{2\pi S_0} \exp\left(-\frac{\nu g |\dot{s}|}{\pi S_0} \right) \quad , \quad \sigma^2_{\dot{S}}(\nu) = \frac{2\pi^2 S_0^2}{\nu^2 g^2} \tag{6}$$

ν is a random variable with pdf $f_\nu(\nu)$, mean $E[\nu]$ and coefficient of variation c_ν. The displacement $S(t)$ is not considered in the state vector as there is no stiffness term involved in this dynamic system. Then, the state vector consists of the velocity $\dot{S}(t)$ and the dummy random process $V(t) = \nu$. This yields $\mathbf{Z}(t)^T = [\dot{S}(t), V(t)]$, thus $N = 1 + 1 = 2$.

5. APPROXIMATING TRANSITIONAL PROBABILITY MATRIX

During a transition from the state $\mathbf{z}_k = (\dot{S}_k, \nu_k)$ at the time t_0, the system is assumed to behave locally Gaussian assuming Δt to be small. The conditional mean value function $\boldsymbol{\mu}(t + \Delta t)$ and conditional covariance matrix $\boldsymbol{\kappa}(t_0 + \Delta t)$ are determined from the differential equations, Köylüoğlu et al. (1995.a).

$$\dot{\boldsymbol{\mu}}(t) = E\Big[\mathbf{a}\big(\mathbf{Z}(t)\big)\Big] \quad , \quad \boldsymbol{\mu}(0) = \mathbf{z}_k \tag{7}$$

$$\dot{\boldsymbol{\kappa}}(t) = E\Big[\frac{\partial}{\partial \mathbf{z}}\mathbf{a}\big(\mathbf{Z}(t)\big)\Big]\boldsymbol{\kappa}(t) + \boldsymbol{\kappa}(t) E\Big[\frac{\partial}{\partial \mathbf{z}}\mathbf{a}\big(\mathbf{Z}(t)\big)\Big]^T + \mathbf{b}(t)\mathbf{b}^T(t) \quad , \quad \boldsymbol{\kappa}(0) = \mathbf{0} \tag{8}$$

(7) and (8) provide the solutions $E[\nu|\mathbf{z}_k] = \nu_k$ and $\kappa_{\dot{S}\nu} = \kappa_{\nu\nu} = 0$, so $q_{\mathbf{Z}}(\dot{s}, \nu, t_0 + \Delta t|\mathbf{z}_k, t_0) = \delta(\nu - \nu_k)\varphi\big((\dot{s} - \mu_{\dot{S}})/\sqrt{\kappa_{\dot{S}\dot{S}}}\big)/\sqrt{\kappa_{\dot{S}\dot{S}}}$, i.e. diffusion only takes place in the $\dot{S}$ direction. $\mu_{\dot{S}}$ and $\kappa_{\dot{S}\dot{S}}$ are obtained by numerical integration of (9) and (10) over the interval $[0, \Delta t]$.

$$\frac{d}{dt}\mu_{\dot{S}}(t) = -\nu_k g\left(1 - 2\Phi\Big(\frac{-\mu_{\dot{S}}(t)}{\sqrt{\kappa_{\dot{S}\dot{S}}(t)}}\Big)\right) \quad , \quad \mu_{\dot{S}}(0) = \dot{S}_k \tag{9}$$

$$\frac{d}{dt}\kappa_{\dot{S}\dot{S}}(t) = -4\nu_k g\sqrt{\kappa_{\dot{S}\dot{S}}(t)}\varphi\Big(\frac{\mu_{\dot{S}}}{\sqrt{\kappa_{\dot{S}\dot{S}}(t)}}\Big) + 2\pi S_0 \quad , \quad \kappa_{\dot{S}\dot{S}}(0) = 0 \tag{10}$$

6. NUMERICAL EXAMPLE

In the numerical example, the conditional variance of the slip velocity conditioned on the mean friction coefficient $\sigma^2_{\dot{S}}(E[\nu])$ is set to 1, i.e., $\frac{2\pi^2 S_0^2}{E[\nu]^2 g^2} = 1$. ν is assumed to be uniformly distributed in the interval $[E[\nu](1-\sqrt{3})c_\nu,\ E[\nu](1+\sqrt{3}c_\nu)]$ where $E[\nu] = 0.5$ and c_ν denotes the coefficient of variation, which will be varied. Path integration results are obtained for the unconditional pdf $f_{\dot{S}}(s)$ of $\dot{S}$ using a mesh with 40×40 cells, extending $\pm 4\sigma_{\dot{S}}$ in the $\dot{S}$ direction and covering the ν direction. The transition time interval is selected as $\Delta t = 0.01\frac{\pi S_0}{E[\nu] g}$. Numerical results shown with full line are compared in Figures 1 and 2 with the exacts ones shown with dashed signature obtained by evaluating the compound probabilities analytically as given in eq. (11). As seen, $\sigma^2_{\dot{S}}$ in (11) is very different than (6). These are equal if $v_c \to 0$.

$$f_{\dot{S}}(\dot{s}) = \int_\nu f_{\dot{S}}(\dot{s}|\nu) f_\nu(\nu) d\nu \quad , \quad \sigma^2_{\dot{S}} = \int_\nu \frac{2\pi^2 S_0^2}{\nu^2 g^2} f_\nu(\nu) d\nu = \frac{1}{1-3c_\nu^2} \frac{2\pi^2 S_0^2}{E[\nu]^2 g^2} \tag{11}$$

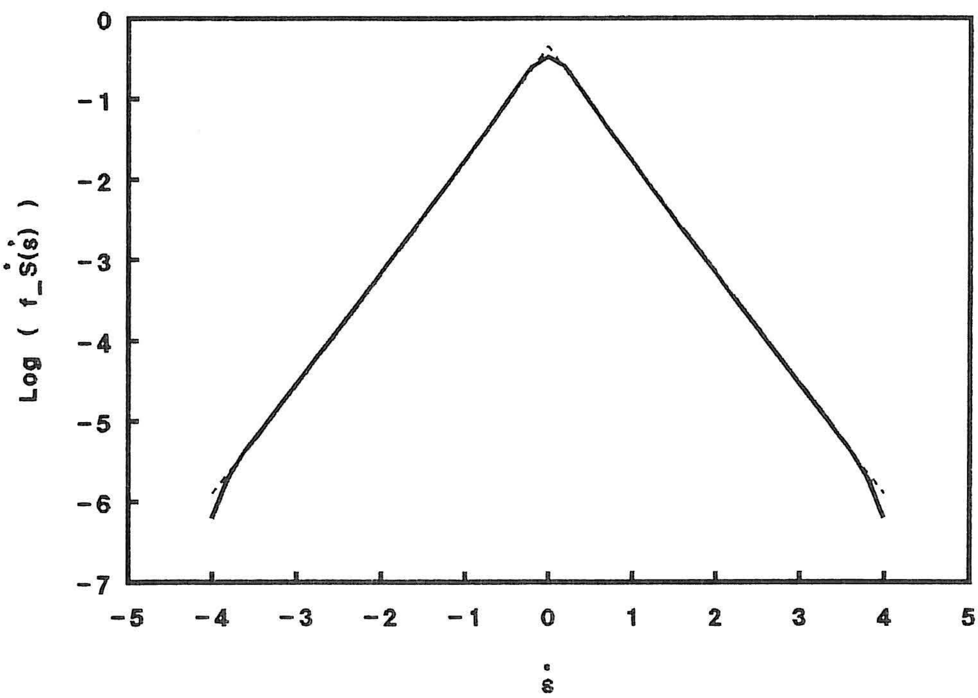

Figure 1: $f_{\dot{S}}(s)$, $c_\nu = 0.1$.

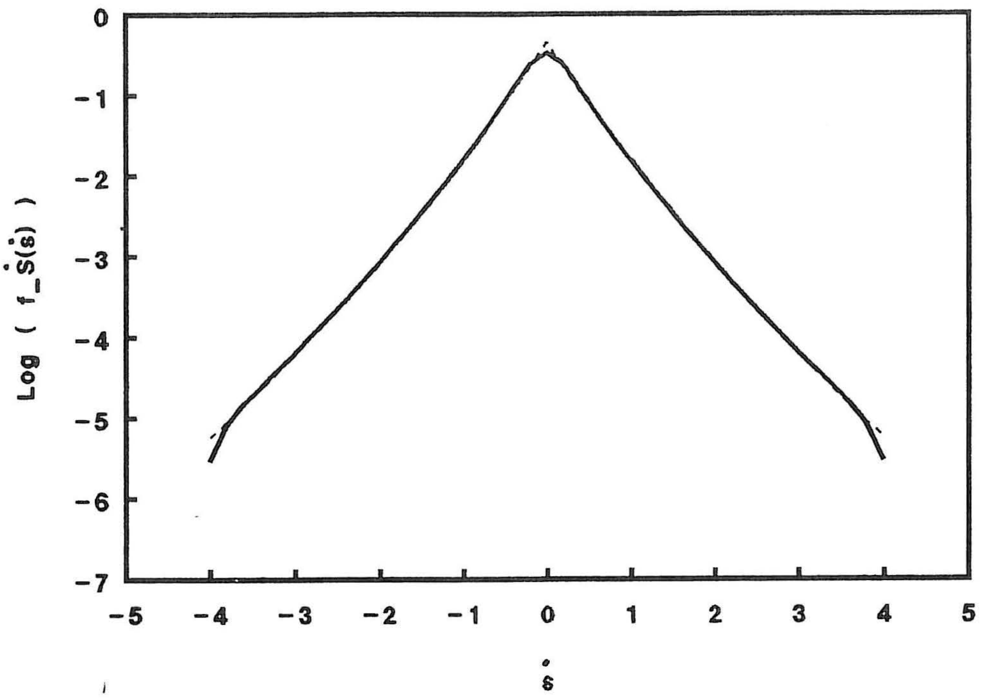

Figure 2: $f_{\dot{S}}(s)$, $c_\nu = 0.3$.

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